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Decision making over necessities through the Choquet integral criterion

Yann Rébillé *

Abstract

Since von Neuman and Morgenstern's (1944) contribution to game theory, a rational decision maker will rank risky prospects according to the celebrated Expected utility criterion.
This method takes lotteries i.e.(simple) probability distributions to represent risky prospects. If the decision maker follows the vN-M axioms (e.g.Kreps (1988)) then there exists a *utility* function such that any probability can be resumed to a lottery having for support the best and the worst state, where the probability that he wins the bet is given by its expected utility.
Probabilities are precise objects to model risk, but the way they do it is incoherent (Dubois and Prade (1988)).
A familiar object in fuzzy set theory is the one of *necessity* or its dual version a *possibility*. In which case the occurence of an event is given by an interval which expresses the imprecision. Nevertheless the description of risk is coherent.
Our concern is to rank different necessity measures and rank them according to the Choquet Expectation criterion (Choquet (1953)).
If the decision maker follows our set of axioms then there exists a *fuzzy set* (Zadeh (1978)) such that any necessity can be resumed to a bet on being perfectly informed of the state which occurs or being totally ignorant, where the degree of information he will get is given by its Choquet expectation.

Keywords: non-additive measures, possibility theory, expected utility.

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1 Introduction

Since Von Neuman and Morgenstern's ([13]) contribution to game theory, a rational decision maker will rank risky prospects according to the celebrated Expected utility criterion.
This method takes lotteries i.e. (simple) probability distributions to represent risky prospects, and the decision maker ranks these probabilities according to a preference relation $\succeq$. Let Ω be a non-empty finite set, and each $\omega \in \Omega$ is called a *state*. Each lottery is written $\sum_{\omega\in\Omega} p(\{\omega\})\delta_\omega$ with $p(\{\omega\}) \geq 0, \sum_{\omega\in\Omega} p(\{\omega\}) = 1$, where δ_ω is the Dirac measure at ω. Let $Prob(\Omega)$ denote the set of loterries on Ω.
If the decision maker follows the vN-M axioms (e.g.[7]) then there exists a function $u : \Omega \longrightarrow [0, 1]$ termed *utility* such that for all probabilities $P, Q \in Prob(\Omega)$

$$P \succeq Q \iff \int udP \geq \int udQ$$

where $\int udP = \sum_{\omega\in\Omega} p(\{\omega\})u(\omega)$. Moreover there are $\omega_1, \omega_0 \in \Omega$ with $u(\omega_1) = 1, u(\omega_0) = 0$ such that for all $P \in Prob(\Omega)$,

$$P \sim (\int udP).\delta_{\omega_1} + (1 - \int udP).\delta_{\omega_0}$$

An interpretation of the last equivalence is that for the decision maker any lottery can be resumed to a bet on head versus tail i.e. a lottery having for support the best and the worst state, where the expected utility of the lottery is interpreted as the probability that he wins the bet.
Probabilities are precise objects to model risk, but the way they do it is incoherent ([6]). Indeed the occurence of an event $A \subset \Omega$ is precisely given by $P(A) = \sum_{\omega\in A} p(\{\omega\})$, but for incomptible events $A, B \subset \Omega, A \cap B = \emptyset$ we might have $P(A), P(B) > 0$.
A familiar object in fuzzy set theory is the one of *necessity* or its dual version a *possibility* ([14]). If v is a necessity then it satisfies for all $A, B \subset \Omega, v(A \cap B) = \mathrm{Min}\{v(A), v(B)\}$ and its dual defined by $v^d(A) = 1 - v(A^c)$ (where $(.)^c$ denotes the complement) satisfies $v^d(A \cup B) = \mathrm{Max}\{v^d(A), v^d(B)\}$.
In which case the occurence of an event A is given by an interval $[v(A), v^d(A)]$ which expresses the imprecision. Nevertheless the description of risk is coherent in the sense that if A, B are incompatible then $v(A), v(B) > 0$ is impossible. This imprecise approach gives a natural interpretation of necessities,

$$\begin{aligned} \forall A, B \subset \Omega, \quad & [v(A \cap B), v^d(A \cap B)] \subset [v(A), v^d(A)] \text{ or } [v(B), v^d(B)] \\ \text{and,} \quad & [v(A \cup B), v^d(A \cup B)] \subset [v(A), v^d(A)] \text{ or } [v(B), v^d(B)] \end{aligned}$$

As an extreme example of necessity is the Dirac measure (which is the sole necessity being also a lottery) which models the perfect knowledge of the state which will occur,

$$\forall A \subset \Omega, \delta_\omega(A) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise} \end{cases}$$

In this case the imprecision interval is minimal,

$$\forall A \subset \Omega, [\delta_\omega(A), \delta_\omega^d(A)] = \begin{cases} \{1\}, & \text{if } \omega \in A \\ \{0\}, & \text{if } \omega \notin A \end{cases}$$

At the other extreme, the decision maker has a perfect ignorance of the state which will occur,

$$\forall A \subset \Omega, u_\Omega(A) = \begin{cases} 1, & \text{if } A = \Omega \\ 0, & \text{otherwise} \end{cases}$$

In this case the imprecision interval is maximal,

$$\forall A \subset \Omega, [u_\Omega(A), u_\Omega^d(A)] = \begin{cases} [0,1], & \text{if } A \neq \emptyset, \Omega \\ \{0\}, & \text{if } A = \emptyset \\ \{1\}, & \text{if } A = \Omega \end{cases}$$

Our concern is to rank different necessity measures on Ω and rank them according to the Choquet Expectation criterion ([2]).
If the decision maker follows our set of axioms then there exists a *fuzzy set* $X : \Omega \longrightarrow [0,1]$ such that for all necessities v, w

$$v \succeq w \iff \int X dv \geq \int X dw$$

Moreover there are $\omega_1, \omega_0 \in \Omega$ with $X(\omega_1) = 1, X(\omega_0) = 0$ such that for all v,

$$v \sim (\int X dv).\delta_{\omega_1} + (1 - \int X dv).u_\Omega$$

An interpretation of the last equivalence is that for the decision maker any necessity can be resumed to a bet on being perfectly informed of the state which occurs or being totally ignorant. The Choquet expectation value of a necessity is interpreted as the degree of information he will get.
The following section introduces the definitions, and the technical material related to Choquet integrals and necessities.
The final section gives some necessary and sufficient conditions on a binary relation on the set of necessities or possibilities in order to obtain a representation of preferences through a Choquet integral.

2 Preliminary results

2.1 Choquet integral

Let S be a nonempty set and $\mathcal{S}$ a non-empty family of subsets of S.
We do not require this family to be an algebra nor to contain the empty set $\emptyset$ or the unit S. Since our work will be devoted to the finite case we will assume from now on that S is finite, thus $\mathcal{S}$ too.

A *set function* v on $\mathcal{S}$ is a real valued function $v : \mathcal{S} \longrightarrow \mathbb{R}$.
v is a *monotone* set function if for all $(A,B) \in \mathcal{S}^2, v(A) \leq v(B)$ whenever $A \subset B$. If $S \in \mathcal{S}$, then v is *normalized* as soon as $v(S) = 1$. If morever $\emptyset \in \mathcal{S}$ then v is a *capacity* as soon as $v(\emptyset) = 0$.

A function $X : S \longrightarrow [0,1]$ is said to be $\mathcal{S}$-*measurable* if for every $t \in (0,1)$, $\{s \in S : X(s) \geq t\}$ belongs to $\mathcal{S}$, as usual for sake of brevety we denote the weak upper level set by $\{X \geq t\}$. $B_{[0,1]}(S,\mathcal{S})$ will denote the set of $\mathcal{S}$-measurable functions taking values in $[0,1]$. For $A \subset S$, $\mathbb{1}_A$ denotes the characteristic function of A, by construction $\mathbb{1}_A$ belongs to $B_{[0,1]}(S,\mathcal{S})$ if and only if $A \in \mathcal{S}$.
This function space $B_{[0,1]}(S,\mathcal{S})$ might not be convex. Assume for instance that $\mathcal{S}$ is not stable by intersection or by union i.e. there are $A, B \in \mathcal{S}$ such that $A \cap B \notin \mathcal{S}$ or $A \cup B \notin \mathcal{S}$.
Let $\alpha \in (0, \frac{1}{2}]$ and $X_\alpha = \alpha.\mathbb{1}_A + (1-\alpha).\mathbb{1}_B$, we have:

$$\{X_\alpha \geq t\} = \begin{cases} A \cup B & \text{, if } t \in (0,\alpha] \\ B & \text{, if } t \in (\alpha, 1-\alpha] \\ A \cap B & \text{, if } t \in (1-\alpha, 1) \end{cases}$$

so $X_\alpha \notin B_{[0,1]}(S,\mathcal{S})$.
If v is a monotone set function defined on $\mathcal{S}$, and X belongs to $B_{[0,1]}(S,\mathcal{S})$, the *Choquet integral* of X with respect to v (see [2]), denoted $\int X dv$, is defined by

$$\int X dv = \int_0^1 v(\{X \geq t\}) dt$$

where the integral under consideration is an improper Riemann integral given by

$$\int_0^1 v(\{X \geq t\}) dt = \lim_{\tau \downarrow 0} \int_\tau^{1-\tau} v(\{X \geq t\}) dt$$

this quantity is well defined since the function $v(\{X \geq .\})$ has finite range on $(0,1)$.
For the case of characteristic function we get: $\forall A \in \mathcal{S}, \int \mathbb{1}_A dv = v(A)$.
Let $X \in B_{[0,1]}(S,\mathcal{S})$, then there is a unique decompostion of X in the following manner $X = \sum_{i=1}^n \alpha_i.\mathbb{1}_{A_i}$, where $\alpha_1, \ldots, \alpha_n > 0$ and $\sum_{i=1}^n \alpha_i \leq 1$ and $A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$ and $A_i \in \mathcal{S}$ for all i where possibly $A_1 = S$ if $S \in \mathcal{S}$. The computation of the Choquet integral of X with respect to v gives,

$$\int X dv = \sum_{i=1}^n \alpha_i v(A_i)$$

By construction as soon as v is monotone the Choquet integral $\int (.) dv$ becomes a monotone functional i.e. $\forall X, Y \in B_{[0,1]}(S,\mathcal{S}), [X \geq Y] \Rightarrow [\int X dv \geq \int Y dv]$.
Let be $X, Y \in B_{[0,1]}(S,\mathcal{S})$, X, Y are said to be *comonotonic* (*compatible*) if for all $(s,t) \in S^2$, $(X(s) - X(t))(Y(s) - Y(t)) \geq 0$.
A fundamental property of the Choquet integral is the one of *comonotonic affinity* (see also [4], [8], [12]): $\forall X, Y \in B_{[0,1]}(S,\mathcal{S})$, $\forall \alpha \in (0,1)$, if $\alpha.X + (1-\alpha).Y \in$

$B_{[0,1]}(S,\mathcal{S})$ and X,Y are comonotonic then $\int \alpha.X+(1-\alpha).Y dv = \alpha.\int X dv + (1-\alpha).\int Y dv$.
We shall see that X,Y comonotonic is a sufficient condition in order that $\alpha.X+(1-\alpha).Y \in B_{[0,1]}(S,\mathcal{S})$ for all $\alpha \in (0,1)$ (see also [9] Proposition 3.3).
In fact we have the following characterization of the functionals which are representable as Choquet integral:

Theorem 2.1 *Let $I : B_{[0,1]}(S,\mathcal{S}) \longrightarrow \mathbb{R}$ be a functional. If I is affine comonotonic and monotone then there exists a monotone set function uniquely defined by $\forall A \in \mathcal{S}, v(A) = I(\mathbb{1}_A)$ such that $I = \int (.)dv$.*
Conversely, if v is monotone set function defined on $\mathcal{S}$ then $\int(.)dv$ is a monotone and affine comonotonic functional.

Proof: For the first part, the set function defined by $\forall A \in \mathcal{S}, v(A) = I(\mathbb{1}_A)$ is clearly monotone and unique.
Let us prove that $I = \int(.)dv$. This is achieved by induction.
Let $X = \sum_{i=1}^{n} \alpha_i.\mathbb{1}_{A_i}$, where $\alpha_1,\ldots,\alpha_n > 0$ and $\sum_{i=1}^{n}\alpha_i \leq 1$ and $A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$ and $A_i \in \mathcal{S}$ for all i.
For the case $n=1$ it is true. Now let $n>1$, we have:

$$\begin{aligned}
I(X) &= I(\sum_{i=1}^{n}\alpha_i.\mathbb{1}_{A_i}) \\
&= I(\alpha_n.\mathbb{1}_{A_n} + (1-\alpha_n)\sum_{i=2}^{n}\frac{\alpha_i}{(1-\alpha_n)}.\mathbb{1}_{A_i}) \\
&= \alpha_n I(\mathbb{1}_{A_n}) + (1-\alpha_n) I(\sum_{i=2}^{n}\frac{\alpha_i}{(1-\alpha_n)}.\mathbb{1}_{A_i}), \quad \text{by affine comonotonicity} \\
&= \alpha_n I(\mathbb{1}_{A_n}) + (1-\alpha_n)\int\sum_{i=2}^{n}\frac{\alpha_i}{(1-\alpha_n)}.\mathbb{1}_{A_i}\, dv, \quad \text{by induction hypothesis} \\
&= \alpha_n v(A_n) + (1-\alpha_n)\sum_{i=2}^{n}\frac{\alpha_i}{(1-\alpha_n)} v(A_i) \\
&= \sum_{i=1}^{n}\alpha_i v(A_i) = \int X dv
\end{aligned}$$

For the converse. It is clear that $\int(.)dv$ is monotone when v is monotone.
We have to prove that $\forall X,Y \in B_{[0,1]}(S,\mathcal{S}), \forall\alpha\in(0,1)$, $\int \alpha.X+(1-\alpha).Y dv = \alpha.\int X dv + (1-\alpha).\int Y dv$ holds as soon as X,Y are comonotonic.
Let $X = \sum_{i=1}^{n}\alpha_i.\mathbb{1}_{A_i}, Y = \sum_{j=1}^{m}\beta_j.\mathbb{1}_{B_j}$ with $A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$, $B_1 \supsetneq \cdots \supsetneq B_m \neq \emptyset$, $\sum_{i=1}^{n}\alpha_i, \sum_{j=m}^{n}\beta_j \leq 1$ and $\alpha_i, \beta_j > 0$.
Since X,Y are comonotonic we have $\forall i \in \{1,\ldots,n\}, j\in\{1,\ldots,m\}$ either $B_j \subset A_i$ or $B_j \supset A_i$ (see [5]). Let $\alpha \in (0,1)$. Define $\{C_k\}_k = \{A_i\}_{i=1}^{n} \cup \{B_j\}_{j=1}^{m}$ and

$$\gamma_k = \begin{cases} \alpha\alpha_i & \text{, if } C_k \in \{A_i\}_{i=1}^{n}, C_k \notin \{B_j\}_{j=1}^{m} \\ (1-\alpha)\beta_j & \text{, if } C_k \in \{B_j\}_{j=1}^{m}, C_k \notin \{A_i\}_{i=1}^{n} \\ \alpha\alpha_i + (1-\alpha)\beta_j & \text{, if } C_k \in \{A_i\}_{i=1}^{n}, C_k \in \{B_j\}_{j=1}^{m} \end{cases}$$

By construction $\sum_k \gamma_k.1\!\!1_{C_k} \in B_{[0,1]}(S, \mathcal{S})$ with $\gamma_k > 0, \sum_k \gamma_k \leq 1$ and $C_k \supsetneq C_{k+1} \neq \emptyset$ and equals $\alpha.X + (1-\alpha).Y$. This gives,

$$\begin{aligned} \int \alpha.X + (1-\alpha).Y \ dv &= \int \sum_k \gamma_k.1\!\!1_{C_k} dv \\ &= \sum_k \gamma_k.v(C_k) \\ &= \alpha \sum_{i=1}^{n} \alpha_i.v(A_i) + (1-\alpha) \sum_{j=1}^{m} \beta_j.v(B_j) \\ &= \alpha \int X \ dv + (1-\alpha) \int Y \ dv \end{aligned}$$

□

2.2 Necessity measures

Our concern is to rank necessity measures, therefore we exhibit a decision criterion based on the Choquet integral to evaluate any necessity. Let Ω be a finite non-empty set. Let S and $\mathcal{S}$ be defined by Ω and $\{A^u : A \neq \emptyset, A \subset \Omega\}$ where $A^u = \{B : A \subset B \subset \Omega\}$ (where A^u stands for the upset generated by A). These sets of subsets of Ω are known as *(principal) filters* (see [3]), we denote the set of filters by $\mathcal{F}(\Omega)$.

A family $\mathcal{F}$ of subsets of Ω is said to be a *filter* if,
(i) $\emptyset \notin \mathcal{F}, \Omega \in \mathcal{F}$,
(ii) $\forall A, B, [A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}]$,
(iii) $\forall A, B, [A \in \mathcal{F}, A \subset B \Rightarrow B \in \mathcal{F}]$.

Now since Ω is a finite set it comes that any filter is *principal* i.e. $\exists A \neq \emptyset \ / \ \mathcal{F} = A^u$. In particular if $A = \{\omega\}$ for some $\omega \in \Omega$ we retrieve an *ultrafilter*.

Proposition 2.1 *Let $v : \mathcal{P}(\Omega) \longrightarrow \mathbb{R}$ be a function. Then v is a necessity if and only if $v \in B_{[0,1]}(\Omega, \mathcal{F}(\Omega))$.*

Proof: (if) Let $v \in B_{[0,1]}(\Omega, \mathcal{F}(\Omega))$, by definition v takes its values in $[0, 1]$.
Let us prove that a $\mathcal{F}(\Omega)$-measurable function is a capacity i.e. normalized monotone set function.
Since for all $\alpha \in (0, 1), \emptyset \notin \{v \geq \alpha\}$ and $\Omega \in \{v \geq \alpha\}$ we have $v(\emptyset) < \alpha \leq v(\Omega)$ thus $v(\emptyset) = 0, v(\Omega) = 1$. So v is a normalized set function.
We now prove that v is monotone.
Let $A \subset B$. Assume $v(A) > v(B)$. Let $\alpha \in (v(B), v(A))$. We have $A \in \{v \geq \alpha\}$ and $B \notin \{v \geq \alpha\}$, so $\{v \geq \alpha\}$ is not a filter.
It remains to check that a $\mathcal{F}(\Omega)$-measurable function is also minitive.
Let $A, B \subset \Omega$ with $v(A) \leq v(B)$.
1^{st} *case*: $v(A) = v(B) = 0$
Let $\alpha \in (0, 1)$, $A, B \notin \{v \geq \alpha\}$ thus $A \cap B \notin \{v \geq \alpha\}$ that is, $v(A \cap B) < \alpha$. So

$v(A \cap B) = 0$.
2^{nd} *case*: $v(A) = 0 < v(B)$
Let $\alpha \in (0, v(B))$, $A \notin \{v \geq \alpha\}$ thus $A \cap B \notin \{v \geq \alpha\}$ that is $v(A \cap B) < \alpha$ so $v(A \cap B) = 0$.
3^{rd} *case*: $0 < v(A) \leq v(B)$
Let $\alpha \in (0, v(A))$, since $A, B \in \{v \geq \alpha\}$ we have $A \cap B \in \{v \geq \alpha\}$ that is $v(A \cap B) \geq \alpha$, and this gives $v(A \cap B) \geq v(A)$. And since v is monotone we get $v(A \cap B) = v(A)$.

(only if) Let v be a necessity and $\alpha \in (0, 1)$.
We have to show that $\{v \geq \alpha\}$ is a filter. Since $v(\emptyset) = 0 < \alpha$, $\emptyset \notin \{v \geq \alpha\}$.
Let $A \in \{v \geq \alpha\}$ and $A \subset B \in \mathcal{P}(\Omega)$, since v is monotone, $\alpha \leq v(A) \leq v(B)$, so $B \in \{v \geq \alpha\}$.
Let $A, B \in \{v \geq \alpha\}$ since v is a minitive set function we get $v(A \cap B) = \min\{v(A), v(B)\} \geq \alpha$, that is $A \cap B \in \{v \geq \alpha\}$.
We can check that $\{v \geq \alpha\}$ is a principal indeed.
Define the following set: $A_0 = \cap_{A \in \{v \geq \alpha\}} A$.
By construction it is clear that for all $A \in \{v \geq \alpha\}$ we have $A_0 \subset A$ thus $\{v \geq \alpha\} \subset A_0^u$. Since $\{v \geq \alpha\}$ is finite, we get: $v(A_0) = \min\{v(A) : A \in \{v \geq \alpha\}\} \geq \alpha$, that is $A_0 \in \{v \geq \alpha\}$, hence $A_0^u \subset \{v \geq \alpha\}$. □

Let us recall that the notion of comonotonicity for necessities has been readily introduced in [10] under the name of *agreement* for minitive set functions :
v, w agree if $\forall A, B \in \mathcal{P}(\Omega)$, $(v(A) - v(B))(w(A) - w(B)) \geq 0$.
A significant characterization can be obtained,

Proposition 2.2 *Let v, w be necessities and $\alpha \in (0, 1)$ then v, w agree if and only if $\alpha.v + (1-\alpha).w$ is a necessity.*

Proof: Let v, w be necessities and $\alpha \in (0, 1)$.
From Proposition 2.1 we have to prove that v, w agree if and only if $\alpha.v + (1-\alpha).w \in B_{[0,1]}(S, \mathcal{S})$.
(if) Assume v, w do not agree, thus there are $A, B \subset \Omega$ s.t. $v(A) > v(B)$ and $w(A) < w(B)$.
Take $t \in (0, 1)$ with $\alpha.v(B) + (1-\alpha).w(A) < t \leq \alpha.v(A) + (1-\alpha).w(A), \alpha.v(B) + (1-\alpha).w(B)$. We have $A, B \in \{\alpha.v + (1-\alpha).w \geq t\}$ but $A \cap B \notin \{\alpha.v + (1-\alpha).w \geq t\}$, so $\{\alpha.v + (1-\alpha).w \geq t\} \notin \mathcal{F}(\Omega)$.
(only if) Assume $\alpha.v + (1-\alpha).w \notin B_{[0,1]}(S, \mathcal{S})$. Since v, w are capacities they take their values in $[0, 1]$ and for all $t \in (0, 1)$, $\{\alpha.v + (1-\alpha).w \geq t\}$ satisfies (i), (iii) of the definition of a filter. Hence there is a $t_0 \in (0, 1)$ such that $\{\alpha.v + (1-\alpha).w \geq t_0\}$ does not fulfill (ii) i.e. $\exists A, B \subset \Omega$ such that

$$\begin{aligned} \alpha.v(A) + (1-\alpha).w(A) &\geq t_0 \\ \alpha.v(B) + (1-\alpha).w(B) &\geq t_0 \\ \alpha.v(A \cap B) + (1-\alpha).w(A \cap B) &< t_0 \end{aligned}$$

So we must have at least one of the following: $v(A\cap B) < v(A)$, $v(A\cap B) < v(B)$, $w(A\cap B) < w(A)$ or $w(A\cap B) < w(B)$.
Assume without loss of generality that $v(A\cap B) < v(A)$ so $v(A\cap B) = v(B) < v(A)$ since v is a necessity. But since
$\alpha.v(B) + (1-\alpha).w(B) \geq t_0 > \alpha.v(A\cap B) + (1-\alpha).w(A\cap B)$ this gives $w(B) > w(A\cap B)$ so $w(B) > w(A)$ since w is a necessity. Hence v, w do not agree. $\square$

Consider now v a necessity defined on $\mathcal{P}(\Omega)$, there is a unique decomposition of v over unanimity games known as the Möbius transform (see [1], [6], [10], [11]) :

$$v = \sum_{i=1}^{n} \alpha_i.u_{A_i}$$

where $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\Omega \supset A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$, and $u_\Omega(A)$ denotes a *unanimity game* i.e. an *elementary belief function* with support A defined by,

$$\forall B \subset \Omega, u_A(B) = \begin{cases} 1, & \text{if } A \subset B \\ 0, & \text{otherwise} \end{cases}$$

or otherwise putted, v can be expressed as follows,
$v = \sum_{i=1}^{n} \alpha_i.\mathbb{1}_{A_i^u}$, where $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^{n} \alpha_i = 1$ and $\emptyset \neq A_1^u \subsetneq \cdots \subsetneq A_n^u$.
As a consequence of Proposition 2.1 and 2.2, given a monotone set function β defined on $\mathcal{F}(\Omega)$ we can compute the Choquet integral of a necessity v with respect to β,

$$\int v d\beta = \sum_{i=1}^{n} \alpha_i.\beta(A_i^u)$$

And for all necessities v, w and $\alpha \in (0,1)$ if v, w agree then,

$$\int \alpha.v + (1-\alpha).w \; d\beta = \alpha \int v d\beta + (1-\alpha) \int w d\beta$$

This object will be the criterion which will be used to rank necessities in order to obtain a *weak integral representation*, that is for all necessities v, w:

$$v \succeq w \iff \int v d\beta \geq \int w d\beta$$

A further look at the Choquet integral of a necessity w.r.t. a monotone set function β can be better elicited if β is better explicitated. One can notice that the family $\mathcal{F}(\Omega)$ possesses a nice property, which is of being stable by intersection. Indeed for $A, B \subset \Omega, \neq \emptyset$ we have: $A^u \cap B^u = (A \cup B)^u$. This property suggests to consider the monotone set function β to be minitive i.e.

$$\forall A, B \subset \Omega, \neq \emptyset, \beta(A^u \cap B^u) = \min\{\beta(A^u), \beta(B^u)\}$$

Since we are dealing with a finite universe the values of β can be fully explicitated. For all $A \subset \Omega, \neq \emptyset$ it holds, $A^u = \cap_{\omega \in A}\{\omega\}^u$, letting $X(\omega) = \beta(\{\omega\}^u)$ we have,

$$\beta(A^u) = \min_{\omega \in A} X(\omega)$$

This last expression is self-advocating and gives for the Choquet integral $\int(.)d\beta$ a *strong integral representation* which is:

$$\begin{aligned}
\int v d\beta &= \sum_{i=1}^{n} \alpha_i.\beta(A_i^u) \\
&= \sum_{i=1}^{n} \alpha_i. \min_{\omega \in A_i} X(\omega) \\
&= \sum_{i=1}^{n} \alpha_i. \int X du_{A_i} \\
&= \int X d(\sum_{i=1}^{n} \alpha_i.u_{A_i}) \\
&= \int X dv
\end{aligned}$$

where $v = \sum_{i=1}^{n} \alpha_i.u_{A_i}$, with $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\Omega \supset A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$. This property established a one to one and onto mapping between *fuzzy set* on Ω and minitive set functions on $\mathcal{F}(\Omega)$ taking values in $[0,1]$, and more particularly a *normalized fuzzy set* obtains (i.e. there are $\omega_1, \omega_0 \in \Omega$ such that $X(\omega_1) = 1, X(\omega_0) = 0$) if and only if β takes value 1 for some ultrafilter on Ω (i.e. $\beta(\{\omega\}^u) = 1$ for some $\omega \in \Omega$) and $\beta(\{\Omega\}) = 0$.

3 Decision model

In this section we provide a simple axiomatization of preferences that can be represented through a Choquet integral. We consider a decision maker who has to rank necessities.

For short we will denote the set of necessities on Ω by $Nec(\Omega)$. A binary relation $\succeq$ on $Nec(\Omega)$ is said to be *complete* if for all $(v, w) \in Nec(\Omega)^2$ we have $v \succeq w$ or $w \succeq v$, *transitive* if for all $(u, v, w) \in Nec(\Omega)^3$ such that $u \succeq v$ and $v \succeq w$ then $u \succeq w$.

A *weak order* $\succeq$ on $Nec(\Omega)$ is a binary relation on $Nec(\Omega)$ which is complete and transitive. As usual we shall write $v \succ w$ for $v \succeq w$ and not$(w \succeq v)$, $v \sim w$ for $v \succeq w$ and $w \succeq v$.

A functional $I : Nec(\Omega) \longrightarrow [0,1]$ *represents* the binary relation $\succeq$ if and only if for all v, w in $Nec(\Omega)$ it holds

$$v \succeq w \iff I(v) \geq I(w)$$

3.1 Weak integral representation of preferences

Now we shall state some axioms that the binary relation $\succeq$ may fulfill.

(WO) $\succeq$ *is a weak order.*
(MON) Monotonicity: $\forall v, w \in Nec(\Omega), [v \geq w] \Rightarrow [v \succeq w]$.

Axiom (WO) is standard. (MON) is quite natural to interprete in term of interval of imprecision:

$$v \geq w \iff \forall A \subset \Omega,\ [v(A), v^d(A)] \subset [w(A), w^d(A)]$$

in which case the decison maker would certainly prefer v over w since the precision is sharper with v then with w.

(AGR) Agreement: $\forall u, v, w, \in Nec(\Omega),\ \forall \alpha \in (0,1)$ *if* u, w *agree and* v, w *agree then* $[u \sim v] \Rightarrow [\alpha.u + (1-\alpha)w \sim \alpha.v + (1-\alpha)w]$.

This axiom expresses the preservation of indifference when we mix a common necessity that agree with both u and v. In particular if we take $w = u_\Omega$, then w agrees with any necessity in which case $\alpha.u + (1-\alpha)u_\Omega \sim \alpha.v + (1-\alpha)u_\Omega$ always holds as soon as $u \sim v$.

(ARCH) $\succeq$ *is Archimedean:* $\forall v, w \in Nec(\Omega)$,
$[v \prec w] \Rightarrow [\exists \alpha \in (0,1) /\ v \prec \alpha.w + (1-\alpha).u_\Omega]$
and,
$[\exists \alpha \in (0,1) /\ \alpha.w + (1-\alpha).u_\Omega \prec v \preceq w] \Rightarrow [\exists \alpha' \in (\alpha, 1) /\ \alpha'.w + (1-\alpha').u_\Omega \preceq v]$

The (ARCH) axiom can be understood in the following manner in conjunction with (MON).
Let $v \prec w$ and $\alpha \in (0,1)$. Since $u_\Omega \leq w$, we have that $u_\Omega \leq \alpha.w + (1-\alpha).u_\Omega \leq w$, under (MON) we get $u_\Omega \preceq \alpha.w + (1-\alpha).u_\Omega \preceq w$, the (ARCH) axiom tells us that if α is close enough to 1 then one should obtain also $v \prec \alpha.w + (1-\alpha).u_\Omega$.
The last axiom ensures that the preference relation is not trivial,

(NDEG) $\succeq$ *is not degenerate:* $\exists v, w \in Nec(\Omega)$ *such that* $v \succ w$

This axiom can be further specified, under (WO) and (MON), (NDEG) is equivalent to: $\exists v \in Nec(\Omega)$ *such that* $v \succ u_\Omega$.
If (AGR) holds too, (NDEG) is simply equivalent to : $\exists \omega \in \Omega$ *such that* $\delta_\omega \succ u_\Omega$.
Assume it does not hold, then $\forall \omega \in \Omega, \delta_\omega \sim u_\Omega$, thus for all $A \subset \Omega, \neq \emptyset$, it holds $u_A \sim u_\Omega$, then applying (AGR) successively gives $\forall v \in Nec(\Omega), v \sim u_\Omega$.

We are now able to state our preference representation theorem,

Theorem 3.1 *Let* $\succeq$ *be a binary relation on* $Nec(\Omega)$, *if* $\succeq$ *satisfies (WO), (MON), (AGR), (ARCH), (NDEG) then there exists a monotone set function* $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that for all* $v, w \in Nec(\Omega)$*:*

$$v \succeq w \iff \int v d\beta \geq \int w d\beta$$

Moreover there is an $\omega_1 \in \Omega$ *such that for all* v *in* $Nec(\Omega)$,

$$v \sim (\int v d\beta).\delta_{\omega_1} + (1 - \int v d\beta).u_\Omega$$

and $\beta(\{\omega_1\}^u) = 1, \beta(\{\Omega\}) = 0$.

Conversely, if the binary relation is represented by a Choquet integral with respect to a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that* $\beta(\{\Omega\}) = 0$ *and* $\beta(\{\omega_1\}^u) = 1$ *for some* $\omega_1 \in \Omega$ *then* $\succeq$ *satisfies (WO), (MON), (AGR), (ARCH), (NDEG).*

Before starting the proof we first state a lemma,

Lemma 3.1 *Let* $\succeq$ *be a binary relation on* $Nec(\Omega)$*, if* $\succeq$ *satisfies (WO), (MON), (ARCH), (NDEG) then there exists a monotone functional* $I : Nec(\Omega) \longrightarrow [0,1]$ *which represents the binary relation* $\succeq$ *that is,*

$$\forall v, w \in Nec(\Omega), v \succeq w \iff I(v) \geq I(w),$$

$$\forall v, w \in Nec(\Omega), v \geq w \Rightarrow I(v) \geq I(w),$$

Moreover there is an $\omega_1 \in \Omega$ *such that,*

$$\forall v \in Nec(\Omega), v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$$

$$\text{and } I(\delta_{\omega_1}) = 1,\ I(u_\Omega) = 0.$$

Proof: Notice first that since Ω is finite there is a $\omega \in \Omega$ such that $\forall \omega \in \Omega, \delta_\omega \preceq \delta_{\omega_1}$. This fact will enable us to bound in term of preference any necessity between u_Ω and δ_{ω_1}.
Let $v \in Nec(\Omega)$. There are $\{\alpha_i, A_i\}_i$ such that $v = \sum_{i=1}^n \alpha_i.u_{A_i}$, with $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^n \alpha_i = 1$, $\Omega \supset A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$. Take $\omega_n \in A_n$, we have $u_\Omega \leq v \leq u_{A_n} \leq \delta_{\omega_n}$ and by (MON) we get $u_\Omega \preceq v \preceq u_{A_n} \preceq \delta_{\omega_n} \preceq \delta_{\omega_1}$.
Using the same argument, (NDEG) entails $u_\Omega \prec v \preceq u_{A_n} \preceq \delta_{\omega_n} \preceq \delta_{\omega_1}$, so $u_\Omega \prec \delta_{\omega_1}$.
We now give a claim:

Claim: $\forall v \in Nec(\Omega), \exists! I(v) \in [0,1] \ / \ v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$

Let us prove now the claim.
Let $L(v) = \{\alpha \in [0,1] : \ \alpha.\delta_{\omega_1} + (1-\alpha).u_\Omega \preceq v\}$.
This set is non-empty and bounded by, since $0 \in L(v)$, thus it admits a supremum. This supremum will give the $I(v)$ we a looking for.
$\bullet L(v)$ is an interval.
Let $0 \leq \alpha < \alpha'$ with $\alpha' \in L(v)$. We have $\alpha.\delta_{\omega_1} + (1-\alpha).u_\Omega \leq \alpha'.\delta_{\omega_1} + (1-\alpha').u_\Omega$ so (MON) entails $\alpha.\delta_{\omega_1} + (1-\alpha).u_\Omega \preceq \alpha'.\delta_{\omega_1} + (1-\alpha').u_\Omega \preceq v$, so $\alpha \in L(v)$.
$\bullet$ Sup $L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega \succeq v$.

Assume Sup $L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega \prec v$ ($\preceq \delta_{\omega_1}$), by (ARCH) $\exists \alpha' \in$ (Sup $L(v), 1)$ such that $\alpha'.\delta_{\omega_1} + (1-\alpha').u_\Omega \preceq v$ that is $\alpha' \in L(v)$ a contradiction.
• Sup $L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega \preceq v$.
Assume Sup $L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega \succ v$, that is Sup $L(v) \notin L(v)$.
By (ARCH) $\exists \alpha \in (0, 1)$ such that $\alpha(\text{Sup } L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega) + (1 - \alpha).u_\Omega \succ v$ that is α Sup $L(v).\delta_{\omega_1} + (1-\alpha \text{ Sup } L(v)).u_\Omega \succ v$ so α Sup $L(v) \notin L(v)$ and α Sup $< \alpha$, but $L(v)$ is an interval so $\forall \beta \in (\alpha \text{ Sup } L(v), \text{Sup } L(v)]$ we have that $\beta \notin L(v)$ so Sup $L(v)$ is not a supremum, a contradiction.

We have proved that Sup $L(v).\delta_{\omega_1} + (1 - \text{Sup } L(v)).u_\Omega \sim v$, thus $L(v)$ admits a maximum. Now let $I(v) = \text{Max } L(v)$ for all $v \in Nec(\Omega)$.
The claim enables us to obtain that:

$$\forall v, w \in Nec(\Omega), v \succ (\prec) w \Rightarrow I(v) > (<) I(w),$$
$$v \geq w \Rightarrow I(v) \geq I(w),$$
$$\text{and } I(\delta_{\omega_1}) = 1,\ I(u_\Omega) = 0.$$

Let $v, w \in Nec(\Omega)$, if $v \geq w$ then by (MON) $v \succeq w$ thus $L(v) \supset L(w)$, so $I(v) \geq I(w)$ i.e. I is a monotone functional.
Let $v, w \in Nec(\Omega), v \succ w$ we have $I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega \succ I(w).\delta_{\omega_1} + (1 - I(w)).u_\Omega$, so by (MON) $I(v) > I(w)$ holds, and the functional I represents $\succeq$.
And the lemma is proved. $\square$

Proof: The converse is immediate.
From Lemma 3.1, there exists a functional I on $Nec(\Omega)$ which is monotone and represents the preference relation.
Moreover there is an $\omega_1 \in \Omega$ such that for all v in $Nec(\Omega)$,

$$v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$$
$$\text{where } I(\delta_{\omega_1}) = 1,\ I(u_\Omega) = 0.$$

Define β through $\beta(A^u) = I(u_A), \forall A \subset \Omega, \neq \emptyset$, thus $\beta(\{\omega_1\}^u) = 1$ and $\beta(\{\Omega\}) = 0$. From our functional representation Theorem 2.1, it remains to establish that I is affine comonotonic, in which case $I = \int (.) d\beta$ will hold and the first part of the theorem will be proved.

Let $v, w \in Nec(\Omega)$ and $\alpha \in (0, 1)$. We have to prove that $I(\alpha.v + (1 - \alpha).w) = \alpha.I(v) + (1 - \alpha).I(w)$ as soon as $\alpha.v + (1 - \alpha).w \in Nec(\Omega)$.
From the claim we have:

$$v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$$

$$w \sim I(w).\delta_{\omega_1} + (1 - I(w)).u_\Omega$$

$$\alpha.v + (1 - \alpha).w \sim I(\alpha.v + (1 - \alpha).w).\delta_{\omega_1} + (1 - I(\alpha.v + (1 - \alpha).w)).u_\Omega$$

Without loss of generality assume $v \succeq w \succ u_\Omega$ holds, thus $I(v) \geq I(w) > 0$.
We have,

$$\begin{aligned} w &\sim I(w).\delta_{\omega_1} + (1 - I(w)).u_\Omega \\ &= \frac{I(w)}{I(v)} I(v).\delta_{\omega_1} + (1 - I(w)).u_\Omega \end{aligned}$$

$$\begin{aligned} &= \frac{I(w)}{I(v)}[I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega] + (1 - \frac{I(w)}{I(v)}).u_\Omega \\ &\sim \frac{I(w)}{I(v)}.v + (1 - \frac{I(w)}{I(v)}).u_\Omega, \text{ by (AGR)} \end{aligned}$$

Applying (AGR) a second time gives

$$\begin{aligned} \alpha.v + (1-\alpha).w &\sim \alpha.v + (1-\alpha).[\frac{I(w)}{I(v)}.v + (1 - \frac{I(w)}{I(v)}).u_\Omega] \\ &= [\alpha + (1-\alpha)\frac{I(w)}{I(v)}].v + (1-\alpha)(1 - \frac{I(w)}{I(v)}).u_\Omega \end{aligned}$$

And since $v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$, we may apply once again (AGR) and this gives,

$$\begin{aligned} \alpha.v + (1-\alpha).w &\sim [\alpha + (1-\alpha)\frac{I(w)}{I(v)}].[I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega] \\ &\quad +(1-\alpha).(1 - \frac{I(w)}{I(v)}).u_\Omega \\ &= [\alpha I(v) + (1-\alpha)I(w)].\delta_{\omega_1} + \{1 - [\alpha I(v) + (1-\alpha)I(w)]\}.u_\Omega \end{aligned}$$

Now since, $\alpha.v + (1-\alpha).w \sim I(\alpha.v + (1-\alpha).w).\delta_{\omega_1} + (1 - I(\alpha.v + (1-\alpha).w)).u_\Omega$. We finally obtain, $I(\alpha.v + (1-\alpha).w) = \alpha I(v) + (1-\alpha)I(w)$.

Now assume $v \succeq w \sim u_\Omega$ holds, thus $I(v) \geq I(w) = 0$.
We have to prove that $I(\alpha.v + (1-\alpha).w) = \alpha I(v)$.
Since $w \sim u_\Omega$ by (AGR) we obtain, $\alpha.v + (1-\alpha).w \sim \alpha.v + (1-\alpha).u_\Omega$,
and since: $v \sim I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega$, we get by (AGR),

$$\begin{aligned} \alpha.v + (1-\alpha).u_\Omega &\sim \alpha.[I(v).\delta_{\omega_1} + (1 - I(v)).u_\Omega] + (1-\alpha).u_\Omega \\ &= \alpha I(v).\delta_{\omega_1} + [\alpha(1 - I(v)) + (1-\alpha)].u_\Omega \\ &= \alpha I(v).\delta_{\omega_1} + (1 - \alpha I(v).u_\Omega \end{aligned}$$

And since, $\alpha.v + (1-\alpha).w \sim I(\alpha.v + (1-\alpha).w).\delta_{\omega_1} + (1 - I(\alpha.v + (1-\alpha).w)).u_\Omega$ we finaly get: $\alpha I(v) = I(\alpha.v + (1-\alpha).w)$. □

3.2 Strong integral representation of preferences

In this subsection we introduce further axioms in order to obtain a representation through a normalized fuzzy set. The key axiom is the following,

(INCL) Inclusion : For all $A, B \subset \Omega, \neq \emptyset$, $[u_A \succeq u_B] \Rightarrow [u_{A\cup B} \sim u_B]$.

Notice that under (WO), the implication in (INCL) can be replaced by an equivalence. Indeed, assume there are A, B such that $u_{A\cup B} \sim u_B$ and $not(u_A \succeq u_B)$. That is $u_{A\cup B} \sim u_B$ and $u_A \prec u_B$ since (WO) holds. Therefore $u_A \preceq u_B$ holds too and this entails $u_{A\cup B} \sim u_A \prec u_B \sim u_{A\cup B}$, which is absurd.

This axiom expresses the idea of a generalized inclusion operation $\dot{\subset}$ defined on $\mathcal{P}(\Omega)/\{\emptyset\}$ through $A\dot{\subset}B \iff u_A \succeq u_B$, since

$$A\dot{\subset}B \iff u_A \succeq u_B \iff u_{A\cup B} \sim u_B \iff A \cup B \doteq B$$

This relation $\dot{\subset}$ becomes a weak order which is compatible with the $\subset$ order relation if (MON) holds, since $A \subset B \iff u_A \geq u_B \Rightarrow u_A \succeq u_B \iff A\dot{\subset}B$.

We can state our preference representation theorem,

Theorem 3.2 *Let $\succeq$ be a binary relation on $Nec(\Omega)$, if $\succeq$ satisfies (WO), (MON), (AGR), (ARCH), (NDEG) and (INCL) then there exists a fuzzy set $X : \Omega \longrightarrow [0,1]$ such that for all $v, w \in Nec(\Omega)$:*

$$v \succeq w \iff \int X dv \geq \int X dw$$

Moreover there are $\omega_1, \omega_0 \in \Omega$ with $X(\omega_1) = 1, X(\omega_0) = 0$ such that for all v in $Nec(\Omega)$,

$$v \sim (\int X dv).\delta_{\omega_1} + (1 - \int X dv).u_\Omega$$

Conversely, if the binary relation is represented by a normalized fuzzy set then $\succeq$ satisfies (WO), (MON), (AGR), (ARCH), (NDEG) and (INCL).

For sake of comparison we recall a version of von Neumann and Morgenstern's theorem ([7]):

Theorem: *Let $\succeq$ be a binary relation on $Prob(\Omega)$, if $\succeq$ satisfies*
(WO): $\succeq$ is a weak order,
(IND): Independance, $\forall P, Q, R \in Prob(\Omega), \forall \alpha \in (0,1), [P \succ Q] \Rightarrow [\alpha.P + (1 - \alpha).R \succ \alpha.Q + (1-\alpha).R]$
(ARCH): $\succeq$ is Archimedean, $\forall P, Q, R \in Prob(\Omega), [P \succ Q \succ R] \Rightarrow [\exists \alpha, \beta \in (0,1) \ / \ \alpha.P + (1-\alpha).R \succ Q \succ \beta.P + (1-\beta).R]$
(NDEG) $\succeq$ is non degenerate, $\exists P, Q \in Prob(\Omega),\ P \succ Q$
if and only if there exists a utility function $u : \Omega \longrightarrow [0,1]$ such that for all probabilities $P, Q \in Prob(\Omega)$

$$P \succeq Q \iff \int u dP \geq \int u dQ$$

Moreover there are $\omega_1, \omega_0 \in \Omega$ with $u(\omega_1) = 1, u(\omega_0) = 0$ such that for all $P \in Prob(\Omega)$,

$$P \sim (\int u dP).\delta_{\omega_1} + (1 - \int u dP).\delta_{\omega_0}$$

Conversely, if the binary relation is represented through an expected utility functional with a utility $u : \Omega \longrightarrow [0,1]$ such that $u(\omega_1) = 1, u(\omega_0) = 0$ for some $\omega_1, \omega_0 \in \Omega$ then $\succeq$ satisfies (WO), (IND), (ARCH), (NDEG).

Notice that the (MON) axiom has been dropped, this is natural since for all $P, Q \in Prob(\Omega)$ if $P \geq Q$ then $P = Q$.

Proof: According to Theorem 3.1 the preference relation has a weak integral representation through a Choquet functional $\int (.)d\beta$, with $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ a monotone set function satisfying $\beta(\{\omega_1\}^u) = 1$ for some $\omega_1 \in \Omega$ and $\beta(\{\Omega\}) = 0$. As noticed in subsection 2.2 we first prove that β is minitive i.e.

$$\forall A, B \subset \Omega, \neq \emptyset, \beta(A^u \cap B^u) = \text{Min}\{\beta(A^u), \beta(B^u)\}$$

Let $A, B \subset \Omega, \neq \emptyset$ such that $u_A \succeq u_B$ by (INCL) it holds $u_{A\cup B} \sim u_B$, which entails:

$$\begin{aligned} \beta(A^u \cap B^u) &= \beta((A \cup B)^u) \\ &= \int u_{A\cup B} d\beta \\ &= \int u_B d\beta \\ &= \beta(B^u) \end{aligned}$$

So β is minitive. Now let $X(\omega) = \beta(\{\omega\}^u)$ for all $\omega \in \Omega$ be the fuzzy set we are looking for.
Since $\beta(\{\Omega\}) = \text{Min}_{\omega\in\Omega}\beta(\{\omega\}^u) = 0$, there is an $\omega_0 \in \Omega$ such that $\beta(\{\omega_0\}^u) = 0$ hence the fuzzy set X is normalized.

For the converse in light of Theorem 3.1 it remains to prove that a preference which has a strong integral representation through $\int X d(.)$ satisfies the inclusion axiom.
Let $A, B \subset \Omega, \emptyset$. Assume $u_A \succeq u_B$ which is equivalent to $\int X du_A \geq \int X du_B$. We have:

$$\begin{aligned} \int X du_{A\cup B} &= \text{Min}_{A\cup B} X \\ &= \text{Min}\{\text{Min}_A X, \text{Min}_B X\} \\ &= \text{Min}\{\int X du_A, \int X du_B\} \\ &= \int X du_B \end{aligned}$$

That is, $u_{A\cup B} \sim u_B$. □

3.3 Decision making over possibility measures

A dual theory can be obtained if instead of ranking necessities a decision maker has to rank possibilities. Let $Poss(\Omega)$ denote the set of possibilities. For $\pi \in Poss(\Omega)$, π^d denotes its *dual* set function of π, defined by $\pi^d(A) = 1 - \pi(A^c), A \subset \Omega$. The correspondance between a possibility and a necessity is one to one since π is a possibility if and only if π^d is a necessity ([6]) and onto since a capacity v is a necessity if and anly if v^d is a possibility. For instance, the dual of a unanimity game u_A is given by:

$$\forall B \subset \Omega, u_A^d(B) = \begin{cases} 1, & \text{if } A \cap B \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

and more particularly:

$$\forall B \subset \Omega, u^d_\Omega(B) = \begin{cases} 1, & \text{if } B \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Moreover the dual transformation preserves the affine combinations:
$\forall v, w \in Nec(\Omega),\ \forall \alpha \in (0,1), [\alpha.v + (1-\alpha).w \in Nec(\Omega)] \Rightarrow [(\alpha.v + (1-\alpha).w)^d = \alpha.v^d + (1-\alpha).w^d \in Poss(\Omega)]$,
the property of agreement: $\forall v, w \in Nec(\Omega), [v, w \text{ agree}] \iff [v^d, w^d \text{ agree}]$,
but the standard setwise order is reversed: $\forall v, w \in Nec(\Omega), v \geq w \iff v^d \leq w^d$.

Starting with a preference relation $\succeq$ on $Poss(\Omega)$ we can define also a preference relation $\succeq^*$ on $Nec(\Omega)$ through,

$$\forall v, w \in Nec(\Omega), v \succeq^* w \iff v^d \succeq w^d$$

The set of axioms introduced for necessities can be translated for possibilities,

(WO) $\succeq$ *is a weak order.*
(ANT) $\succeq$ *is Antitone:* $\forall \pi, \rho \in Poss(\Omega), [\pi \geq \rho] \Rightarrow [\pi \preceq \rho]$.
(AGR) Agreement: $\forall \pi, \rho, \sigma \in Poss(\Omega),\ \forall \alpha \in (0,1)$ *if* π, σ *agree and* ρ, σ *agree then* $[\pi \sim \rho] \Rightarrow [\alpha.\pi + (1-\alpha).\sigma \sim \alpha.\rho + (1-\alpha)\sigma]$.
(ARCH) $\succeq$ *is Archimedean:* $\forall \pi, \rho \in Poss(\Omega)$,
$[\pi \prec \rho] \Rightarrow [\exists \alpha \in (0,1)/\ \pi \prec \alpha.\rho + (1-\alpha)u^d_\Omega]$
and,
$[\exists \alpha \in (0,1)/\ \alpha.\pi + (1-\alpha)u^d_\Omega \prec \rho] \Rightarrow [\exists \alpha' \in (\alpha,1)/\ \alpha'.\pi + (1-\alpha').u^d_\Omega \prec \rho]$.
(NDEG) $\succeq$ *is not degenerate:* $\exists \pi, \rho \in Poss(\Omega)$ *such that* $\pi \succ \rho$.

In which case a weak integral representation of $\succeq$ is achieved via a Choquet integral,

$$\forall \pi, \rho \in Poss(\Omega), \quad \pi \succeq \rho \iff \int \pi^d d\beta \geq \int \rho^d d\beta$$

and

$$\forall \pi \in Poss(\Omega), \quad \pi \sim (\int \pi^d d\beta).\delta_{\omega_1} + (1 - \int \pi^d d\beta).u^d_\Omega$$

In order to obtain a strong integral representation we adapt the axiom of inclusion,

(INCL) Inclusion : For all $A, B \subset \Omega, \neq \emptyset$, $[u^d_A \succeq u^d_B] \Rightarrow [u^d_{A \cup B} \sim u^d_B]$

and retrieve a representation through a normalized fuzzy set,

$$\forall \pi, \rho \in Poss(\Omega), \quad \pi \succeq \rho \iff \int X d\pi^d \geq \int X d\rho^d \iff \int Z d\pi \leq \int Z d\rho$$

and

$$\forall \pi \in Poss(\Omega), \quad \pi \sim (1 - \int Z d\pi).\delta_{\omega_1} + (\int Z d\pi).u^d_\Omega$$

where X is the normalized fuzzy set obtain via theorem 3.2 and Z is a normalized fuzzy set defined by $Z = 1 - X$.

This statement holds since we can prove that $\int X d\pi^d = 1 - \int (1-X) d\pi$ holds for any fuzzy set. And this gives a natural extension to fuzzy sets of the duality equation: $\forall A \subset \Omega, \pi^d(A) = 1 - \pi(A^c)$.

Proof: Let $\pi^d = \sum_{i=1}^n \alpha_i . u_{A_i}$, with $\alpha_1, \ldots, \alpha_n > 0$, $\sum_{i=1}^n \alpha_i = 1$, $\Omega \supset A_1 \supsetneq \cdots \supsetneq A_n \neq \emptyset$. We have,

$$\begin{aligned}
\int X d\pi^d &= \int X d(\sum_{i=1}^n \alpha_i . u_{A_i}) \\
&= \sum_{i=1}^n \alpha_i \int X du_{A_i} \\
&= \sum_{i=1}^n \alpha_i \min_{A_i} X \\
&= 1 + \sum_{i=1}^n \alpha_i \min_{A_i} (X-1) \\
&= 1 - \sum_{i=1}^n \alpha_i \max_{A_i} (1-X) \\
&= 1 - \sum_{i=1}^n \alpha_i \int (1-X) du_{A_i}^d \\
&= 1 - \int (1-X) d(\sum_{i=1}^n \alpha_i u_{A_i}^d) \\
&= 1 - \int (1-X) d\pi
\end{aligned}$$

□

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